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Simultaneous Approximation and Birkhoff Interpolation II: The Periodic Case

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I. INTRODUCTION

We continue [6] by studying the simultaneous approximation of a real-valued function and its derivatives on the unit circle K. In order to do this, we will need a theorem giving sufficient conditions for the solvability of the Birkhoff interpolation problem in the periodic case.

Let $C_i(K)$ be the space of i-times continuously differentiable real-valued functions defined on the unit circle K. If $\|\cdot\|$ denotes the uniform norm on $C(K)$, then

$$\|f\|_k = \max_{j=0,1,\ldots,k} \|f^{(j)}\|$$

will be called the simultaneous norm. Here $f^{(j)}$ denotes the jth derivative of f in the clockwise direction.

Let H be a finite-dimensional Chebycheff subspace of $C_k(K)$. We will investigate whether $f \in C_k(K)$ has a unique best approximation from H with respect to $\|\cdot\|_k$. This is usually not the case and we will determine the precise dimension of the set $\Omega_H(f)$ of best simultaneous approximations to f from H.

The ideas are similar to those in [6] but, due to the periodic nature of the functions involved, the structure of $\Omega_H(f)$ is particularly simple. In fact, it will be shown that the set of best approximations consists of either one point or is one-dimensional. This is in contrast to the algebraic case [6], in which the dimension of $\Omega_H(f)$ could have been as high as k if the approximation was with respect to $\|\cdot\|_k$.

Let $A \subset C_j(K)$. $A^{(j)}$ will then denote the set

$$A^{(j)} = \{a^{(j)} \mid a \in A\}.$$

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Our main theorem will be

THEOREM 1. *Let H be an N-dimensional subspace of $C_{k+2}(K)$ such that each non-zero $h \in H^{(k+2)}$ has at most $N-1$ zeros. If $f \in C_{k+1}(K)$, then $\Omega_{H,k}(f)$, the set of best approximations to f from H with respect to $\|\cdot\|_k$, has dimension at most one.*

II. THE PERIODIC BIRKHOFF INTERPOLATION PROBLEM

In order to prove Theorem 1, we will need sufficient conditions for the regularity of the Birkhoff interpolation problem for periodic Chebycheff subspaces H of dimension N which satisfy the additional condition that no element of $H^{(k)}$ can have more than $N-1$ zeros. These assumptions on H are more restrictive than one might assume at first glance, for

LEMMA 2. *Let H be an N-dimensional subspace of $C_k(K)$ such that each non-zero $h \in H^{(k)}$ has at most $N-1$ zeros. Then*

(1) *H is a Chebycheff space;*

(2) *N is odd;*

(3) $\dim H^{(i)} = N-1$, $i = 1, 2, \dots, k$;

(4) *each non-zero $h \in H^{(i)}$ has at most $N-1$ zeros, $i = 1, 2, \dots, k$;*

(5) *the constant function belongs to H.*

Proof. A very important, although elementary, fact about zeros of periodic functions is

LEMMA 3. *If $f \in C_1(K)$ has N zeros, then f' also has at least N zeros.*

This is Rolle's theorem for periodic functions and explains why results for periodic systems differ so much from results for systems defined on intervals. Parts (1) and (4) of Lemma 2 follow from Lemma 3 immediately, for if any non-zero $f \in H^{(i)}$, $i = 0, 1, \dots, k-1$, had N or more zeros, then by repeated application of Lemma 3, $f^{(k-i)}$, which belongs to $H^{(k)}$, would also have at least N zeros contradicting the assumption on $H^{(k)}$.

From (4) now follows that no non-zero $f \in H^{(1)}$ can have more than $N-1$ zeros. If the dimension of $H^{(1)}$ were N, then $H^{(1)}$ would be a Chebycheff space and therefore contains a strictly positive function (see [3] or [7]). But this is impossible since if $f \in H^{(1)}$, then $f = h'$ for some $h \in H$ and

$$\int_0^{2\pi} f(t)\, dt = \int_0^{2\pi} h'(t)\, dt = h(2\pi) - h(0) = 0.$$

This is incompatible with $f(t)>0$ for all $t\in[0, 2\pi)$. Thus dim $H^{(1)}\leqslant N-1$. Since the constant function is the only continuous solution to $h'=0$ on K, it follows that the constant function belongs to H and that dim $H^{(1)}= N-1$. Since the constant function is also the only continuous solution to $f^{(i)}=0$ on K, it also follows that dim $H^{(i)}=N-1$, $i=1,..., k$.

We have shown that H is a Chebycheff subspace of dimension N. Then N is necessarily odd since there exist no Chebycheff subspaces of even dimension on K (see [7]).

Our theorem thus applies to a fairly restricted class of subspaces. One of these is the subspace T_n of trigonometric polynomials. This was shown by Johnson [2]. Another example is the space of periodic algebraic polynomials $\mathscr{P}_{n,r}$, which we present here for the first time

$$\mathscr{P}_{n,r}=\{P\in \Pi_n \mid P^{(i)}(0)=P^{(i)}(2\pi),\ i=0, 1,..., r\}.$$

Π_n denotes the space of algebraic polynomials of degree not exceeding n. The proof that $\mathscr{P}_{n,r}$ is indeed an $n-r$ dimensional subspace of $C_r(K)$ which satisfies the conditions of Theorem 1 will be deferred to the end of our exposition.

The main tool used to prove Theorem 1 will be the theory of Birkhoff interpolation for periodic systems. The method of proof follows the same lines as for the trigonometric case as developed in [1].

Let $E=(e_{ij})$ be an $m\times(n+1)$ matrix consisting of only zeros and ones. E is called an incidence matrix. We will number the rows (which correspond to knots of interpolation) from 1 to m and the columns (which correspond to the derivative being interpolated) from 0 to n. Let $H\subset C_n(K)$. The Birkhoff interpolation problem (B.I.P.) for E and H is to find an element $h\in H$ satisfying

$$h^{(j)}(t_i)=b_{ij}$$

for given data b_{ij}, knots $0\leqslant t_1<\cdots<t_m<2\pi$ and for those pairs (i, j) such that $e_{ij}=1$. If there is a unique solution to this problem for each set of data and each choice of ordered knots, E is said to be regular (with respect to H). Otherwise E is singular. A sequence of E of length l is a maximal sequence of ones which has length l and lies in some row of E: $e_{i,j}=0$, $e_{i,j+1}=\cdots=e_{i,j+l}=1$, $e_{i,j+l+1}=0$. The possibilities $j=-1$ and $j+l+1=n+1$ are allowed. A sequence is called even or odd if it has even or odd length. The sequence given above is said to be (periodically) supported if some $e_{s,t}=1$ for $t\leqslant j$.

THEOREM 4. *Let E be an $m\times(n+1)$ incidence matrix which has exactly N ones, has at least one* 1 *in its* 0th *column and which has no odd supported sequences. Suppose that $H\subset C_{n+1}(K)$ has dimension N and that no*

$g \in H^{(n+1)}$ has more than $N-1$ zeros without vanishing identically. Then E is regular with respect to H. That is, the B.I.P. corresponing to E always has a unique solution in H.

Moreover, if E satisfies the above conditions but has more than N ones, the only solution to the homogeneous B.I.P. is the zero function.

Proof. We will show that the only solution to the homogeneous B.I.P. is the zero function. This implies that a unique solution to the B.I.P. with arbitrary data exists.

Let m_i be the number of ones in the ith column of E and let $T = \{t_1, ..., t_m\}$ be given. Suppose that $h \in H$ satisfies the homogeneous B.I.P. for E. Then h has m_0 zeros. By Rolle's theorem, unless $h \equiv 0$, h' has a sign change between each of the m_0 zeros of h. We would like to add to these, the m_1 zeros prescribed by E to conclude that h' has $m_0 + m_1$ zeros. But some of the m_0 zeros determined by Rolle's theorem may coincide with some of the m_1 zeros determined by E. If this is the case, we must analyse more carefully what happens. Then there are three knots $t_i < t_j < t_k$ such that $e_{i,0} = e_{j,1} = e_{k,0} = 1$ and $e_{l,0} = 0$ for $i < l < k$. Thus a supported begins at position $(j, 1)$. This sequence is, by assumption, even. But h' must change its sign between t_i and t_k. Therefore, either h' has another zero between t_i and t_k, or the zero t_j must have a higher multiplicity. In the first case, we do not lose a zero when counting the zeros of h'. In the second case, we may conclude that h' has a zero of multiplicity at least one more than the length of the sequence (inasfar as h is sufficiently often differentiable). As we will be repeating this process, we will have lost a zero of h' but we will regain it at some higher derivative. Since such a coincidence can occur at most once for each sequence, the lost zero will be rewon at the latest when we have arrived at the $n+1$st derivative.

Thus by repeating this process, we may conclude that $h^{(n+1)}$ has at least $m_0 + m_1 + \cdots + m_n = N$ zeros. Thus $h^{(n+1)} \equiv 0$. By Lemma 2, dim $H^{(i)} = N-1$ for $i = 1, 2, ..., n+1$. Therefore $h' \equiv 0$ and so $h \equiv$ constant. But $h(t) = 0$ at one of the knots, thus $h \equiv 0$ as desired.

If one assumes a little more about the zeros of the elements of $H^{(n)}$, then no assumption on $H^{(n+1)}$ is needed. The concept we use is that of the multiplicity of zeros, which may be found in Karlin and Studden [3]. Let $f \in C(K)$ or $f \in C(I)$, where I is the closed interval $[a, b]$. Let $f(t) = 0$. We say that f has a zero of order one at $t \in K$ or $t \in [a, b]$ if f changes sign at t or if t is one of the endpoints of I. We say that f has a zero of multiplicity two if $f(s) \geqslant 0$ or $f(s) \leqslant 0$ in some neighborhood of t. $\tilde{Z}(f)$ will denote the number of zeros of f counting multiplicities in this way.

It is well known that if $H \subset C(K)$ or $H \subset C(I)$ is a Chebycheff subspace of degree n, then $\tilde{Z}(f) \leqslant n-1$ for each $f \in H$. Moreover, for any $f \in C(K)$, $\tilde{Z}(f)$ is always an even number. The interesting feature of these considerations is that a zero t of f may be counted with multiplicity two even though $f'(t)$ may not exist.

THEOREM 5. *Let E be an $m \times (n+1)$ incidence matrix which has exactly N ones, has at least one 1 in its 0th column and which has no odd supported sequences. Suppose that $H \subset C_n(K)$ has dimension N and that no $g \in H^{(n)}$ has more than $N-1$ zeros counting multiplicities. Then E is regular with respect to H.*

If E satisfies the above conditions and has more than N ones, then the only solution to the homogeneous B.I.P. corresponding to E is the zero function.

The proof follows that of Theorem 4 except that one may only conclude that an h satisfying the homogeneous B.I.P. has an nth derivative which has at least N zeros counting multiplicities. The main new technical difficulty occurs when, while counting zeros, a new zero is to be added to a sequence ending in the nth column. A satisfactory treatment of this difficulty may be found in Keener [4].

A particularly simple case in which the hypothesis is satisfied is if $G = \text{span}\,\{1, H^{(n)}\}$ is a Chebycheff space of dimension N (if $\dim H = N$). Then no element of G and hence no element of $H^{(n)}$ can have more than $N-1$ zeros counting multiplicities. It will be shown later that this holds for $\mathscr{P}_{n,r}$ the set of periodic algebraic polynomials. There are some hints that this favorable circumstance is not improbable. Firstly, for any H satisfying the conditions of Theorem 5, the constant function can never belong to $H^{(n)}$ so that $\dim \text{span}\{1, H^{(n)}\} = N$ always. Secondly, the following lemma, whose proof follows easily from Lemma 3, holds.

LEMMA 6. *Let H be an N-dimensional subspace of $C_n(K)$ such that no non-zero element of $H^{(n)}$ has more than $N-1$ zeros. Then span$\{1, H^{(i)}\}$ is an N-dimensional Chebycheff subspace of $C_{n-i}(K)$ for $i = 1,\ldots, n-1$.*

III. SIMULTANEOUS APPROXIMATION

We may use these results on the periodic B.I.P. to characterize the dimension of the set of best approximations to a function with respect to a simultaneous norm.

We defined

$$\|f\|_k = \max_{j=0,1,\ldots,k} \|f^{(j)}\|$$

for $f \subset C_k(K)$. Let $H \subset C_k(K)$. Then we denote by $\Omega_{H,k}(f)$ (or by $\Omega_H(f)$), the set of best approximations to f from H with respect to $\|\cdot\|_k$ and by

$E_{H,k}(f)$ the degree of approximation to f from H. Then for each $h \subset \Omega_H(f)$, $\|f-h\|_k = E_{H,k}(f)$. By $U_j(f, h)$, we denote the extremal sets of the approximation h to f

$$U_j(f, h) = \{t \in K \mid \|f^{(j)} - h^{(j)}\| = \|f-h\|_k\}$$

for $j = 0, 1, ..., k$. The extremal sets are compact and at least one of them is non-empty. Since $\Omega_H(f)$ is convex, it has a relative interior and a relative dimension. $\dim \Omega_H(f)$ will denote the relative dimension of $\Omega_H(f)$. Any element of H lying in the relative interior of $\Omega_H(f)$ will be called a minimal best approximation (to f). This terminology is motivated by

LEMMA 7. *If h is a minimal best approximation to f, then for any other best approximation g,*

$$U_j(f, h) \subset U_j(f, g), \qquad j = 0, 1, ..., k.$$

Thus the extremal sets of a minimal best approximation are the smallest possible. It follows immediately that for any two minimal best approximations h, g to f, we have $U_j(f, h) = U_j(f, g)$, $j = 0, 1, ..., k$. These common sets will be denoted by $U_j(f)$ and will be called the extremal sets of f since they depend only on f and H and not on the choice of a minimal best approximation.

LEMMA 8. *We have*

$$U_j(f) \cap U_{j+1}(f) = \varnothing, \qquad j = 0, 1, ..., k-1$$

and if $g, h \in \Omega_H(f)$, $f \in C_{k+1}(K)$ and $H \subset C_{k+1}(K)$, then

$$g^{(j)}(t) = h^{(j)}(t), \qquad t \in U_j(f)$$
$$g^{(j+1)}(t) = h^{(j+1)}(t), \qquad t \in U_j(f)$$

for all $j = 0, 1, ..., k$.

The fact that two best approximations must coincide on so many points yields the uniqueness theorem.

THEOREM 9. *Let H be an N-dimensional subspace of $C_{k+2}(K)$ such that no non-zero element of $H^{(k+2)}$ has more than $N-1$ zeros. Let $f \in C_{k+1}$. If $U_0(f) \neq \varnothing$, then the best approximation to f from H with respect to $\|\cdot\|_k$ is unique. If $U_0(f) = \varnothing$, then* $\dim \Omega_{H,k}(f) = 1$. *In any case,* $\dim \Omega_{H,k}(f) \leqslant 1$.

Proof. Let h be a minimal best approximation to f and g be any other element of $\Omega_H(f)$. By Lemma 8,

$$(h-g)^{(j)}(t)=0$$
$$(h-g)^{(j+1)}(t)=0$$

for $t \in U_j(t)$, $j=0, 1,..., k$. Let E be the incidence matrix corresponding to these conditions. We will show that E contains at least N ones. By Lemma 8, we know that the pairs of conditions appearing above cannot overlap each other. Thus E has only even sequences. If therefore E had $M \leqslant N-1$ ones, we could add $N-M$ new ones to the 0th column (adding new knots if necessary) to obtain an incidence matrix $\bar{E}$ satisfying the conditions of Theorem 4. It follows that there is a $u \in H$ satisfying

$$u^{(j)}(t)=\sigma(h(t)-f(t))$$

for $t \in U_j(t)$, $j=0, 1,..., k$, where $\sigma(t)$ is the sign of t. But then $h-\varepsilon u$, for all $\varepsilon>0$ sufficiently small, is a better approximation to f than h is. This is by assumption not possible, so $M \geqslant N$.

As in the proof of Theorem 4, $(h-g)^{(k+1)}$ has M zeros and therefore $(h-g)^{(k+1)} \equiv 0$. Since $\dim H^{(1)} = \dim H^{(k+1)}$, $(h-g)' \equiv 0$ and so $h-g$ is a constant. If $U_0(f) \neq \varnothing$, then $(h-g)(t)=0$ at one of the knots which implies that $h-g \equiv 0$ and that $\Omega_{H,k}(f)=\{h\}$. If $U_0(f)=\varnothing$, then $h+c \in \Omega_H(f)$ for all constants c such that $|c|$ is sufficiently small.

THEOREM 10. *Let H be an N-dimensional subspace of $C_{k+1}(K)$ such that no non-zero element of $H^{(k+1)}$ can have more than $N-1$ zeros counting multiplicities. Let $f \in C_{k+1}(K)$. If $U_0(f) \neq \varnothing$, then the best approximation to f from H with respect to $\|\cdot\|_k$ is unique. If $U_0(f)=\varnothing$, then* $\dim \Omega_{H,k}(f)=1$. *In any case,* $\dim \Omega_{H,k}(f) \leqslant 1$.

IV. AN EXAMPLE

By $\mathscr{P}_{n,r}$, we denote the space of periodic algebraic polynomials:

$$\mathscr{P}_{n,r}=\{P \in \Pi_n \mid P^{(i)}(0)=P^{(i)}(2\pi), \quad i=0, 1,..., r\}$$

We will show that if $n-r$ is odd, then $\mathscr{P}_{n,r}$ satisfies the assumptions of Theorem 10 with $N=n-r$ and for any $k \leqslant r-1$.

LEMMA 11. *Let $\mathscr{P}_{n,r}$ be the space of periodic algebraic polynomials defined above. Suppose $n-r$ is odd.*

Then

(1) $\dim \mathscr{P}_{n,r} = n - r$;

(2) $\mathscr{P}_{n,r}$ *is a Chebycheff subspace of* $C_r(K)$;

(3) *for each* i, $0 \leqslant i \leqslant r$, *no element of* $\mathscr{P}^{(i)}_{n,r}$ *can have more than* $n - r - 1$ *zeros counting multiplicities.*

Proof. We will first show that if $n - r$ is odd, then no element of $\mathscr{P}^{(i)}_{n,r}$ can have more than $n - r - 1$ zeros counting multiplicities for $i = 0, 1,..., r$. Due to Lemma 3, it suffices to show this for $i = r$; i.e., to show that for each $P \in \mathscr{P}^{(r)}_{n,r}$, $\tilde{Z}(P) \leqslant n - r - 1$. The elements of $\mathscr{P}^{(r)}_{n,r}$ are polynomials P of degree not exceeding $n - r$ and for which $P(0) = P(2\pi)$. Suppose that some polynomial P had $\tilde{Z}(P) \geqslant n - r$. Since $n - r$ is odd, $\tilde{Z}(P) \geqslant n - r + 1$ since $\tilde{Z}(P)$ is always even. We will now consider P as being defined on $I = [0, 2\pi]$. Let $\tilde{Z}_K(P)$ (respectively $\tilde{Z}_I(P)$) be the number of zeros of P counting multiplicities when defined on K (respectively when defined on $I = [0, 2\pi]$). It can easily be seen that $\tilde{Z}_K(P) \leqslant \tilde{Z}_I(P)$. But $\tilde{Z}_K(P) \geqslant n - r + 1$ so that also $\tilde{Z}_I(P) \geqslant n - r + 1$. Since $P \in \Pi_{n-r}$, $P \equiv 0$. This proves (3).

By definition, $\mathscr{P}_{n,r}$ is a subspace consisting of those elements of the $n + 1$-dimensional space Π_n which satisfy $r + 1$ linear conditions. Thus $\dim \mathscr{P}_{n,r} \geqslant n - r$. But, if $n - r$ is odd, we have just shown that no element of $\mathscr{P}_{n,r}$ can have more than $n - r - 1$ zeros. Therefore $\mathscr{P}_{n,r}$ is a Chebycheff subspace and $\dim \mathscr{P}_{n,r} = n - r$ which proves (1).

Since, by definition, all elements of $\mathscr{P}_{n,r}$ have periodic derivatives of order up to r, (2) has also been shown.

An interesting aspect of this proof is that absolutely no calculations were needed to show that $\dim \mathscr{P}_{n,r} = n - r$. An alternative proof would have been to show that the $r + 1$ linear conditions determining $\mathscr{P}_{n,r}$ are linearly independent (which is not hard to do). Then $\dim \mathscr{P}_{n,r} = n - r$ follows immediately (even if $n - r$ is even).

COROLLARY 12. *Let* $f \in C_{k+1}(K)$. *Let* $n \geqslant r \geqslant 0, n - r$ *be odd and* $k \leqslant r - 1$. *Then the dimension of the set of best approximations from* $\mathscr{P}_{n,r}$ *to* f *with respect to* $\|\cdot\|_k$ *does not exceed* 1. *In particular, if* $U_0(f) \neq \varnothing$, *the best approximation is unique. If* $U_0(f) = \varnothing$, *the set of best approximations has dimension exactly one.*

COROLLARY 13. *For* $1 \leqslant i \leqslant r$,

$$\operatorname{span}\{1, \mathscr{P}^{(i)}_{n,r}\} = \mathscr{P}_{n-i,r-i}.$$

V. Concluding Remarks

As in [6], the norm of simultaneous approximation could have been chosen to be

$$\|f\|_F = \max_{i=1,\dots,p} \|f^{(k_i)}\|$$

where $0 \leqslant k_1 < k_2 < \cdots < k_p$. For this semi-norm, the formulation of Theorems 9 and 10 and their proofs remain the same.

The assumption that $f \in C_{k+1}(K)$, which was made in Theorems 9 and 10, appears unnatural since the norm $\|\cdot\|_k$ involves derivatives of order only up to k. The theorems are, however, false without this assumption as can be seen by example given in [2].

Theorems 4 and 5 on the regularity of the periodic B.I.P. are not sharp of course because they only give necessary conditions for regularity. By comparing these general theorems with their trigonometric counterparts [1], one sees which of the assumptions are indispensable.

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